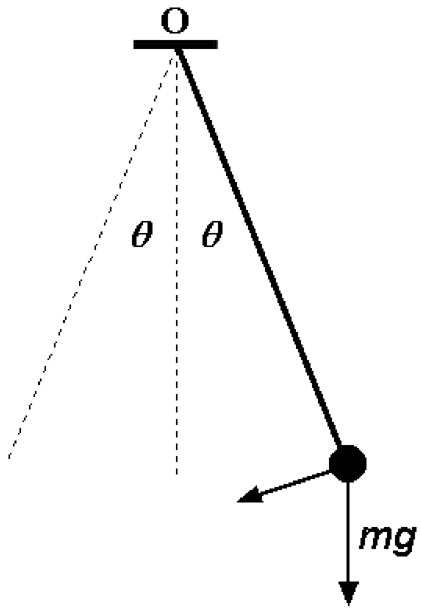
- ***H***

$$H = K + V = \dot{x}^2 / 2 + \omega_0^2 \cos x, \;\; \dot{p} = -\partial H / \partial q, \; \dot{q} = \partial H / \partial p$$

$$x = angle \;\&\; \omega_0 = intrinsic \;\; frequency, p = \dot{x}, q = x$$

-

$$\partial H / \partial t = 0 \Rightarrow \ddot{x} + \omega_0^2 \sin x = 0$$

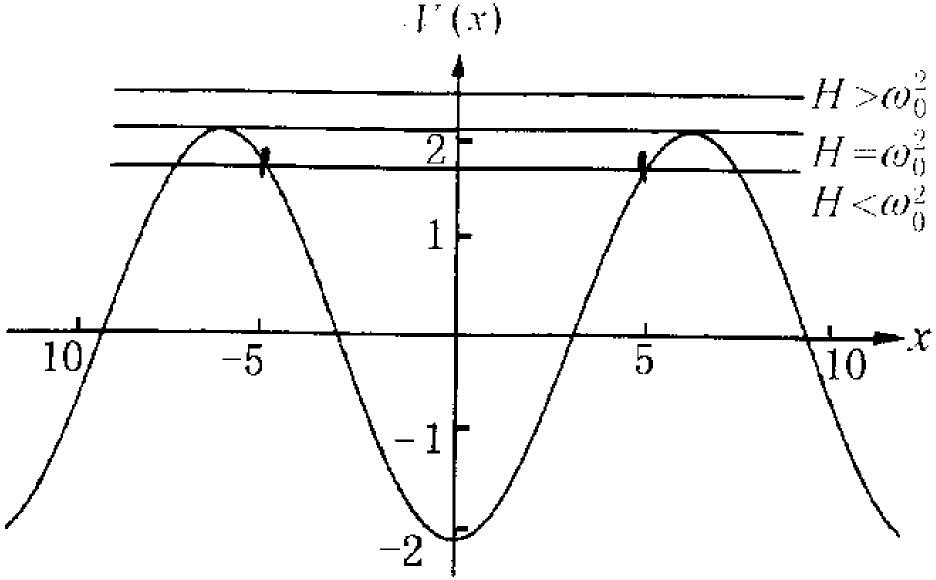


$\theta = x$

$$\dot{x}_e = 0,\ x_e = n\pi(n = 0, \pm 1, \pm 2, \ldots)$$

$$V(x) = -\omega_0^2 \cos x \Rightarrow \begin{cases} V_{min} = -\omega_0^2,\ x_e = n\pi(n \geq 0) \\ V_{max} = \omega_0^2,\ x_e = n\pi(n \leq 0) \end{cases}$$

•



• ***H=ω₀²***

$$x_e = -n\pi (n \geq 1),\ \dot{x}_e = 0$$

$$\dot{x}^2 / 2 = \omega_0^2 \cos x + \omega_0^2$$

$$\dot{x} = \pm 2\omega_0 \cos(x/2)$$

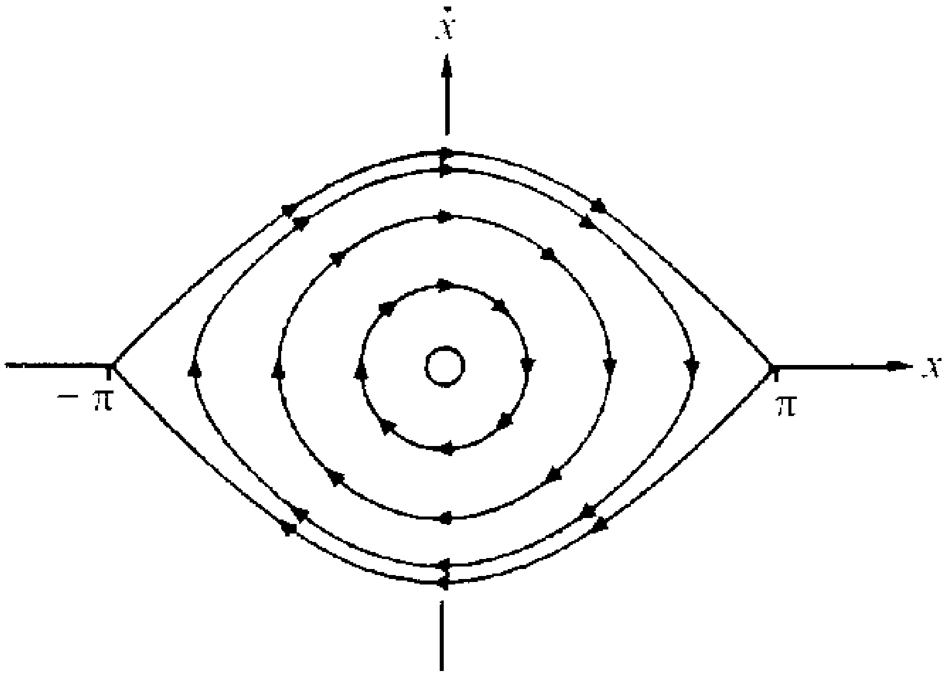
$$for\ \ t = 0, x = 0$$

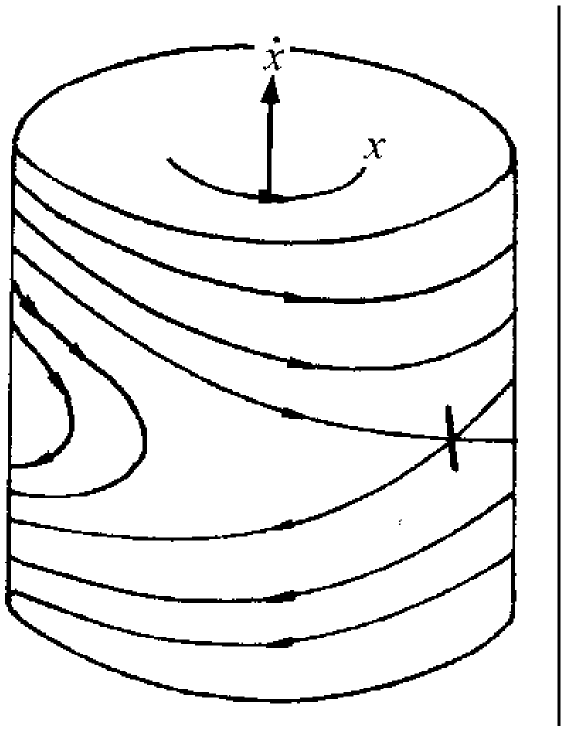
$$x = 4arctan(e^{\omega_0 t}) - \pi$$

- $H<\omega_0^2$

  $H>\omega_0^2$

- $x_0 \Rightarrow \pi$ $H>\omega_0^2$





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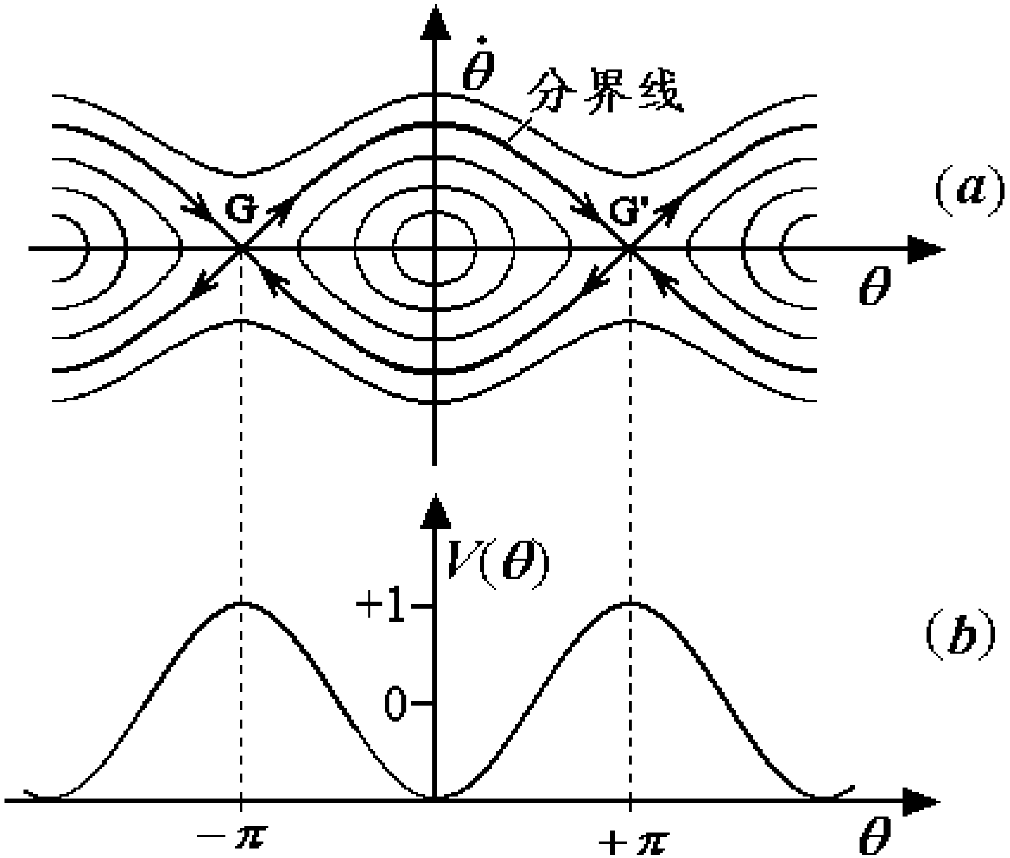
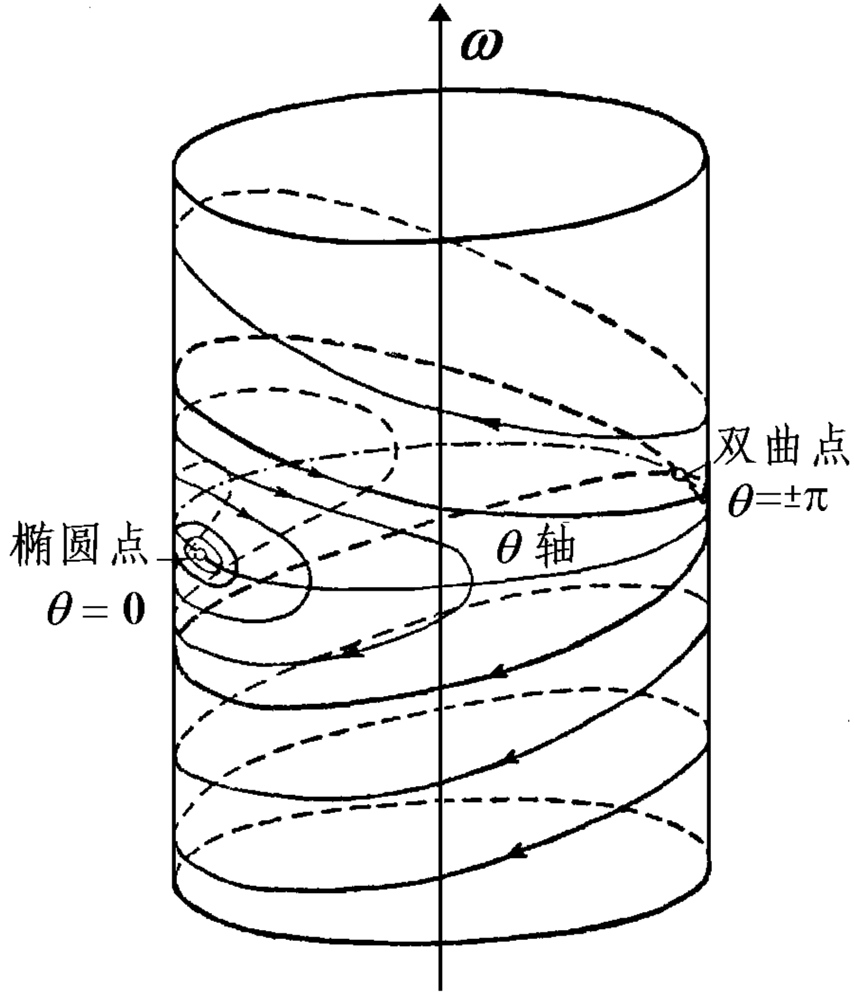


$\dot{\theta}$
分界线
(a)
G
G'
θ
$V(\theta)$
+1
0
(b)
$-\pi$
$+\pi$
θ



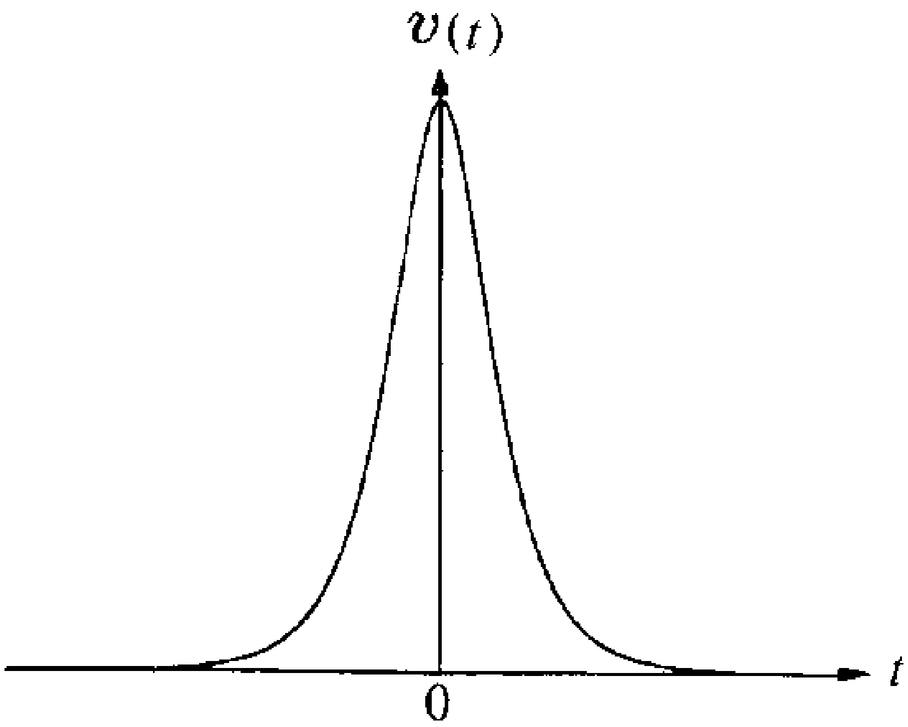
ω
双曲点
$\theta=\pm\pi$
椭圆点
$\theta=0$
θ 轴

•

$$\because \sin 2x = \frac{2\tan x}{(1+\tan^2 x)} \quad \therefore \cos\frac{x}{2} = \frac{1}{\cosh(\omega_0 t)}$$

$$x = 4\arctan(e^{\omega_0 t}) - \pi \quad \therefore \dot{x} = v(t) = \pm\frac{2\omega_0}{\cosh(\omega_0 t)}$$

• **(Soliton)**



$$\dot{x} = v(t) = \pm \frac{2\omega_0}{\cosh(\omega_0 t)}$$

-

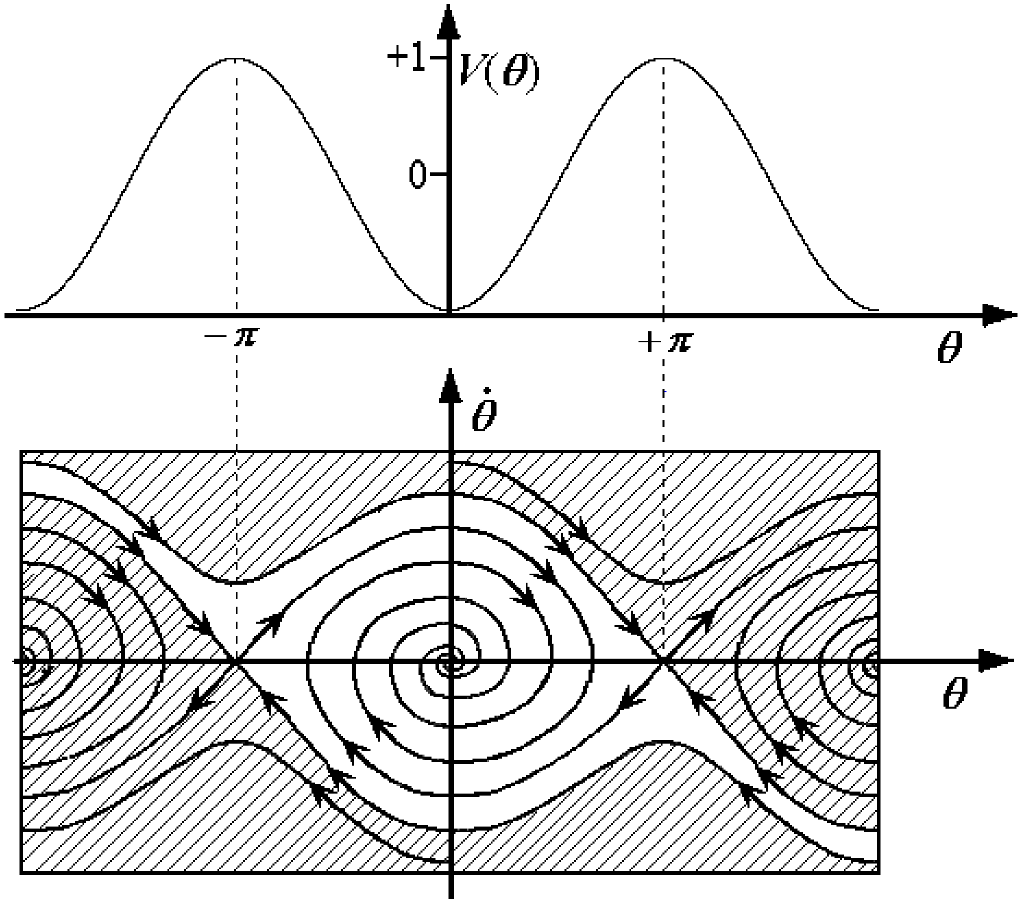
$$\ddot{\theta} + 2\beta\dot{\theta} + \omega_0^2\theta = 0 \qquad \lambda^2 + 2\beta\lambda + \omega_0^2 = 0$$

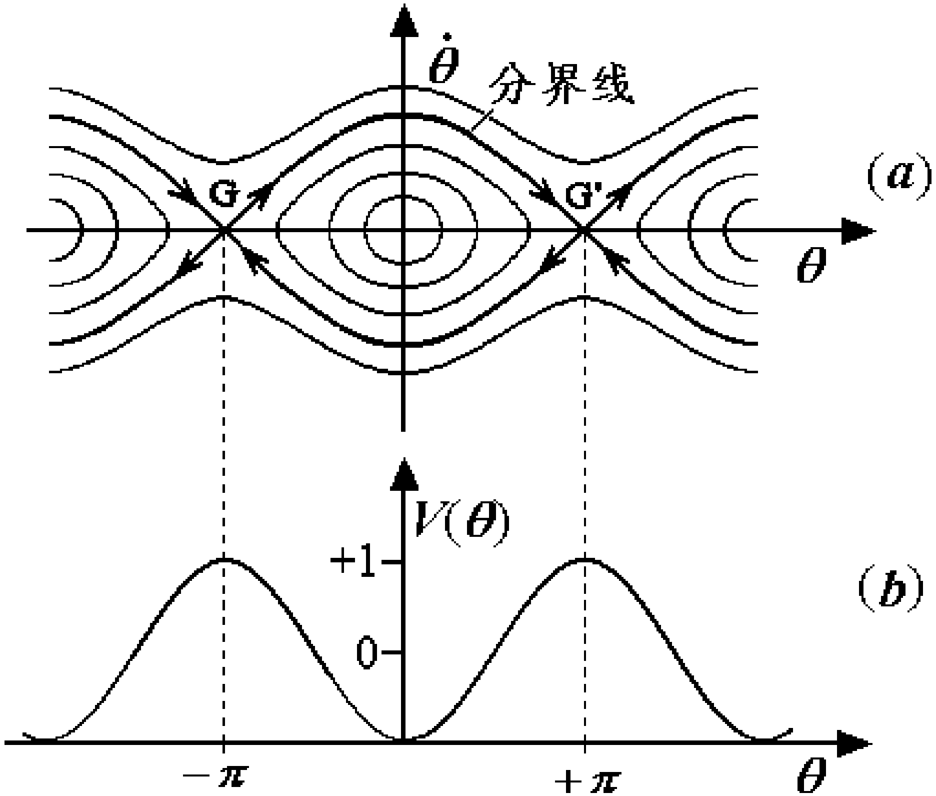
$$\lambda_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \qquad \omega = \sqrt{\omega_0^2 - \beta^2}$$

$$\lambda_{1,2} = -\beta \pm i\omega$$

$$\theta = P \cdot e^{-\beta} \cos(\omega t + \varphi)$$

$$\dot{\theta} = -P \cdot e^{-\beta t}[\beta \cos(\omega t + \varphi) + \omega \sin(\omega t + \varphi)]$$


+1
V(θ)
0
−π
+π
θ
θ̇


θ̇
分界线
G
G'
θ
(a)
V(θ)
+1
0
(b)
−π
+π
θ

-

-

$$\ddot{x} = F$$

$$v = \dot{x}(t)$$

- ***x(t)*** ***v*** ***(x,y)***

(x,y)

•

$$\begin{cases} \dot{x} = v_x = f(x, y) \\ \dot{y} = v_y = g(x, y) \end{cases}$$

$$\begin{cases} \dot{x} = v_x = f(x, y, z) \\ \dot{y} = v_y = g(x, y, z) \\ \dot{z} = v_z = h(x, y, z) \end{cases}$$

• ***t*** ***t***

-

$$\ddot{x} + x = A\cos(\omega t) \Leftrightarrow \begin{cases} \dot{x} = y \\ \dot{y} = -x + A\cos(z) \\ \dot{z} = \omega \end{cases}$$

$$\begin{cases} \dot{u} = f(u, v, t) \\ \dot{v} = g(u, v, t) \end{cases} \Leftrightarrow \begin{cases} \dot{u} = f(u, v, w) \\ \dot{v} = g(u, v, w) \\ \dot{w} = 1 \end{cases}$$

- t

$$\begin{cases} \ddot{x} + x = 0 \\ \ddot{x} + \sin x = 0 \end{cases}$$

$$\dot{x}\ddot{x} + x\dot{x} = 0 \Rightarrow$$

$$H(x,y) = \frac{\dot{x}^2}{2} + \frac{x^2}{2} = \frac{x^2}{2} + \frac{y^2}{2} = Const.$$

- ***H***

$$\begin{cases} u = \dot{x} = y \equiv \dfrac{\partial H(x,y)}{\partial y} \\ v = \dot{y} = -x \equiv -\dfrac{\partial H(x,y)}{\partial x} \end{cases}$$

$$\vec{v} = (u,v,w) \Rightarrow div\vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{1}{V}\frac{dV}{dt}$$

- ***divv*=0** **V** ***divv*<0**

-

$$\begin{cases} \ddot{x} + x = 0 \\ \ddot{x} + \sin x = 0 \end{cases}$$

$$div\vec{v} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} = \frac{\partial^2 H}{\partial x \partial y} + \frac{\partial^2 H}{\partial x \partial y} = 0$$

-

$$\ddot{x} + \alpha\dot{x} + x = 0 \quad \begin{cases} \dot{x} = y \\ \dot{y} = -\alpha y - x \end{cases} \Rightarrow div\vec{v} = -\alpha < 0$$

-

-

( )

$$
\begin{cases}
\dot{x} = f(x,y,z) = 0 \\
\dot{y} = g(x,y,z) = 0 \\
\dot{z} = h(x,y,z) = 0
\end{cases}
$$

-

$$\dot{x} = f(x) = \mu x(1-x)$$

- $x^*=0$ $x^*=1$
- μx μx^2

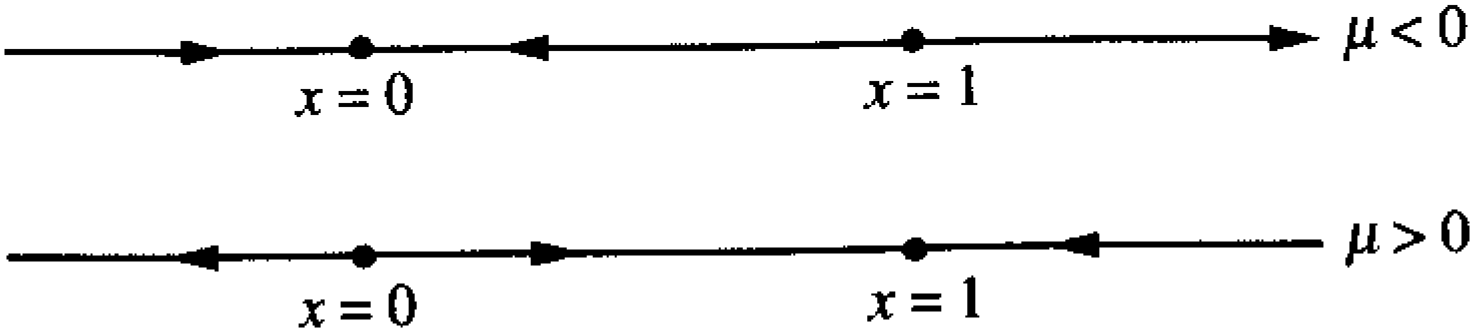
$$\delta\dot{x} = \left.\frac{\partial f}{\partial x}\right|_{x=x^*} \delta x \qquad \delta x = \delta x_0 e^{\lambda t}, \quad \lambda = \left.\frac{\partial f}{\partial x}\right|_{x=x^*}$$

- λ $x=x^*$
- $Re\lambda>0$ δx t
- $Re\lambda<0$ δx t

•

$$\lambda = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \mu \Leftrightarrow \lambda = \left.\frac{\partial f}{\partial x}\right|_{x=1} = -\mu$$

• **(μ=0)**



•

$$\begin{cases} \dot{x} = f(x,y) = ax + by \\ \dot{y} = g(x,y) = cx + dy \end{cases}$$

• **(0, 0)**

$$\ddot{x} - (a+d)\dot{x} - (bc - ad)x = 0$$

$$\ddot{x} + p\dot{x} + qx = 0 \Leftarrow p = -(a+d), \quad q = -(bc - ad)$$

• ( )

-

$$div\vec{v} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} = (a + d) = -p < 0$$

$$\lambda^2 + p\lambda + q = 0$$

-

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(0,0)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

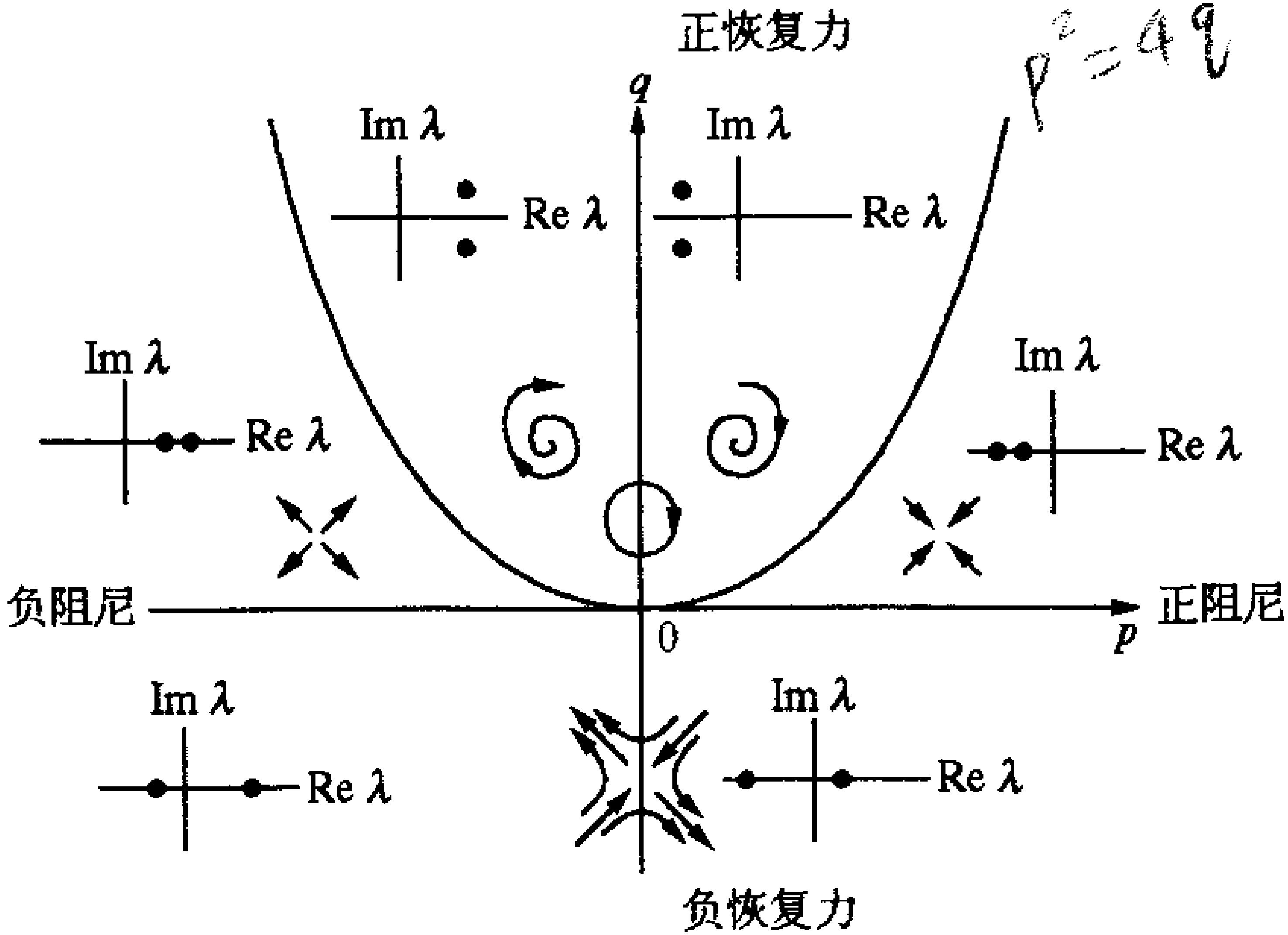
-

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \qquad \lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

-

$p^2=4q$

-



正恢复力
p²=4q
q
Im λ
Re λ
Im λ
Re λ
Im λ
Re λ
Im λ
Re λ
负阻尼
正阻尼
0
p
Im λ
Re λ
Im λ
Re λ
负恢复力

- **(*p*, *q*)**

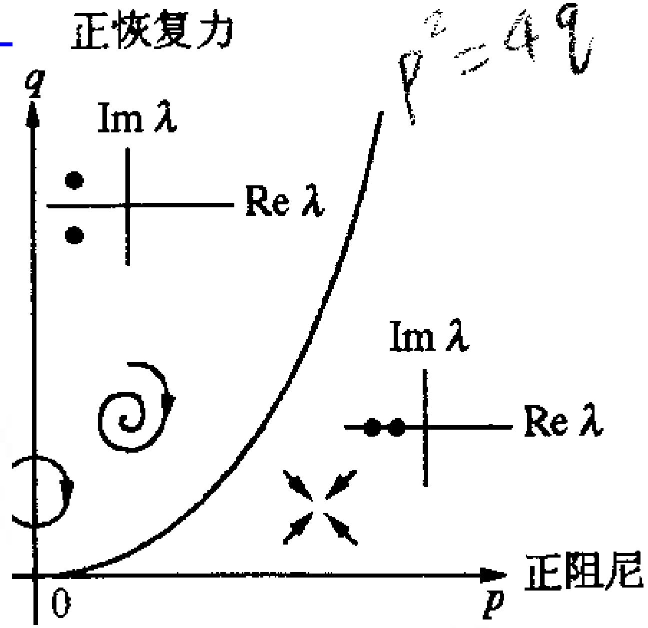
  + **p**

  $p^2=4q$

  λ

$$\delta x = \delta x_0 e^{\lambda t} = \delta x_0 e^{(Re\,\lambda + i\,Im\,\lambda)t}$$

$$= \delta x_0 e^{(Re\,\lambda)t}\left[cos(Im\,\lambda)t + i\,sin(Im\,\lambda)t\right]$$



- **Reλ<0** **(0, 0)** **$p^2<4q$**

  (focus)

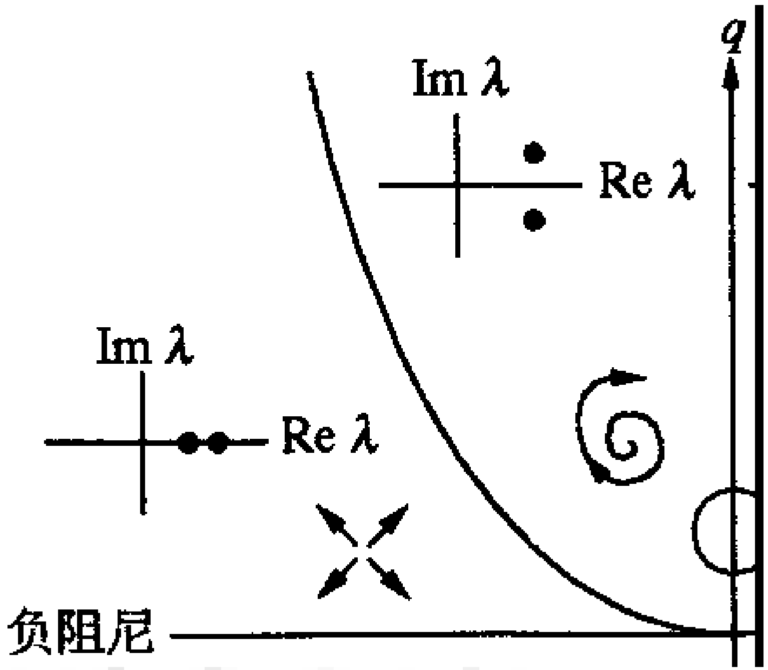
- **$p^2>4q$** **λ** **(0, 0)** **(node)**

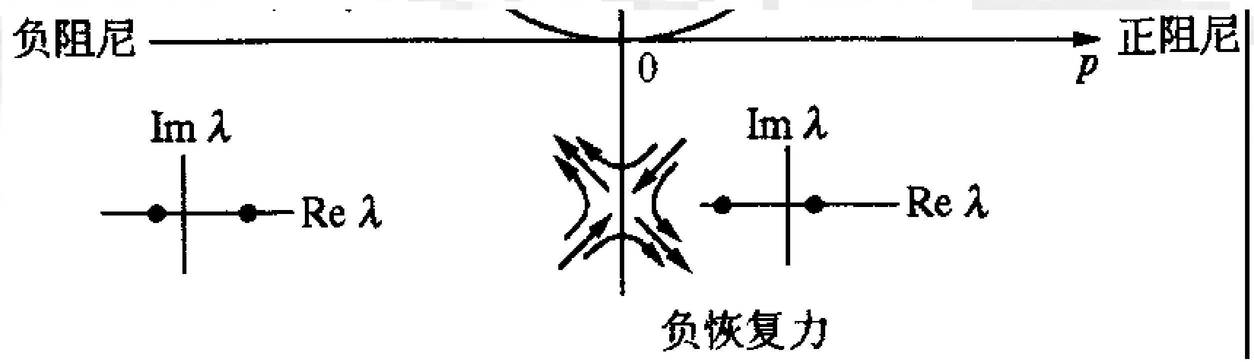
- (p, q) +

  $p^2=4q$ (1) λ

  ($|p|$) (2) λ

  ($|p|$) (0, 0)



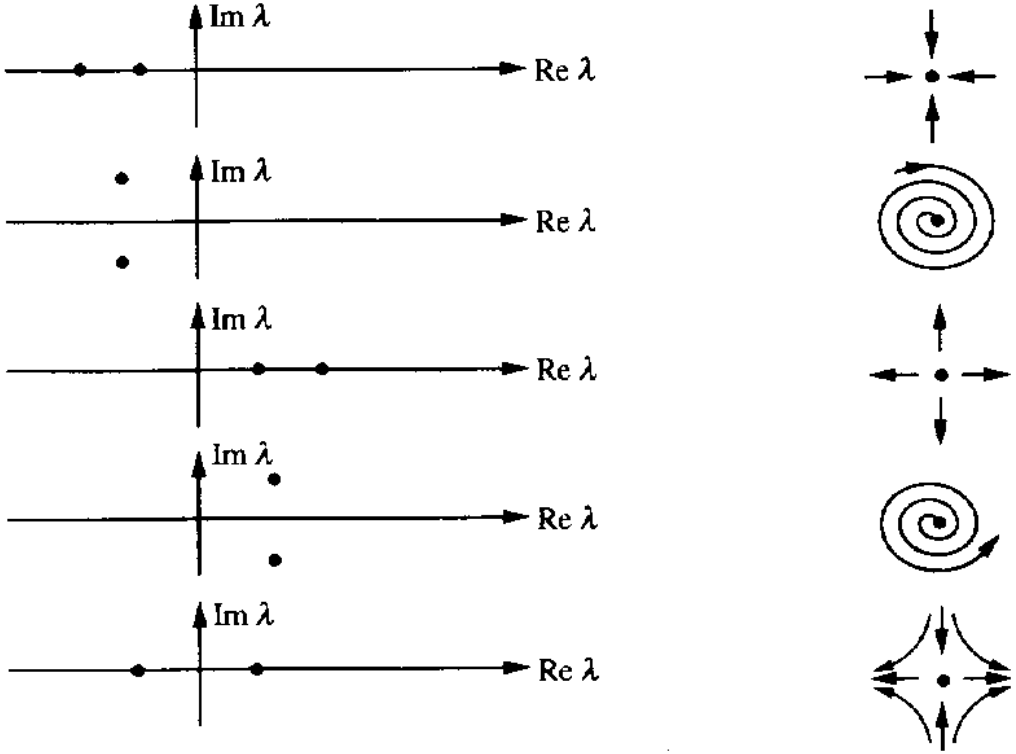


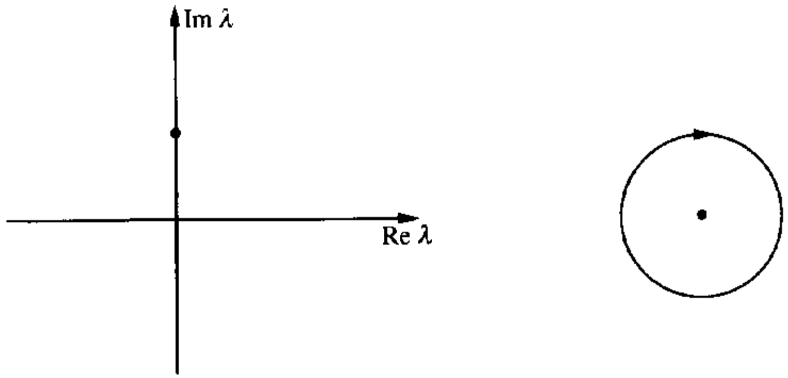
- $q<0$ λ

  (0, 0) (saddle)

- (p, q) $p=0$ λ $(0, 0)$ $(0, 0)$

-





-

$$\begin{cases} \dot{x} = v_x = f(x, y, z) \\ \dot{y} = v_y = g(x, y, z) \\ \dot{z} = v_z = h(x, y, z) \end{cases}$$

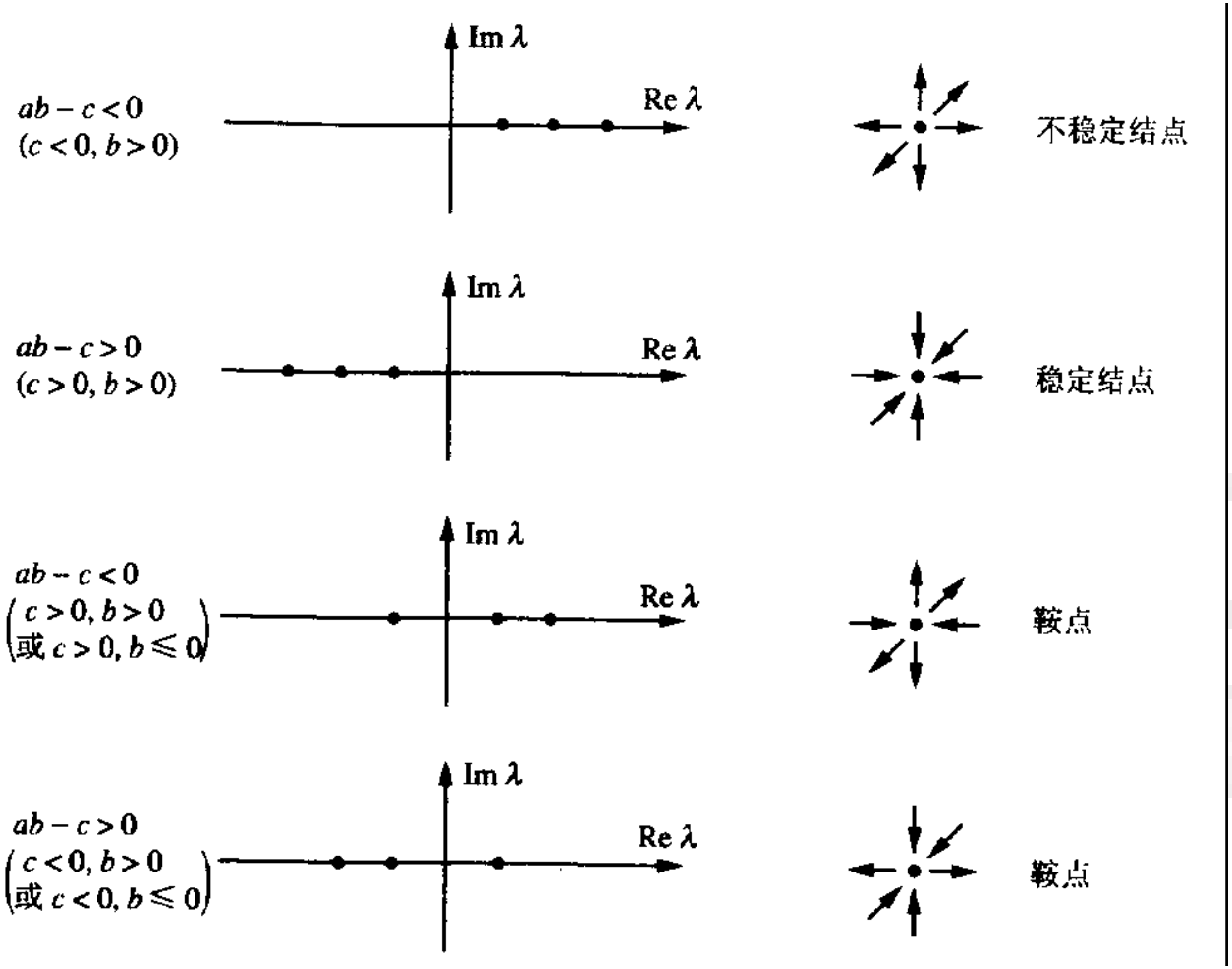
-

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0$$

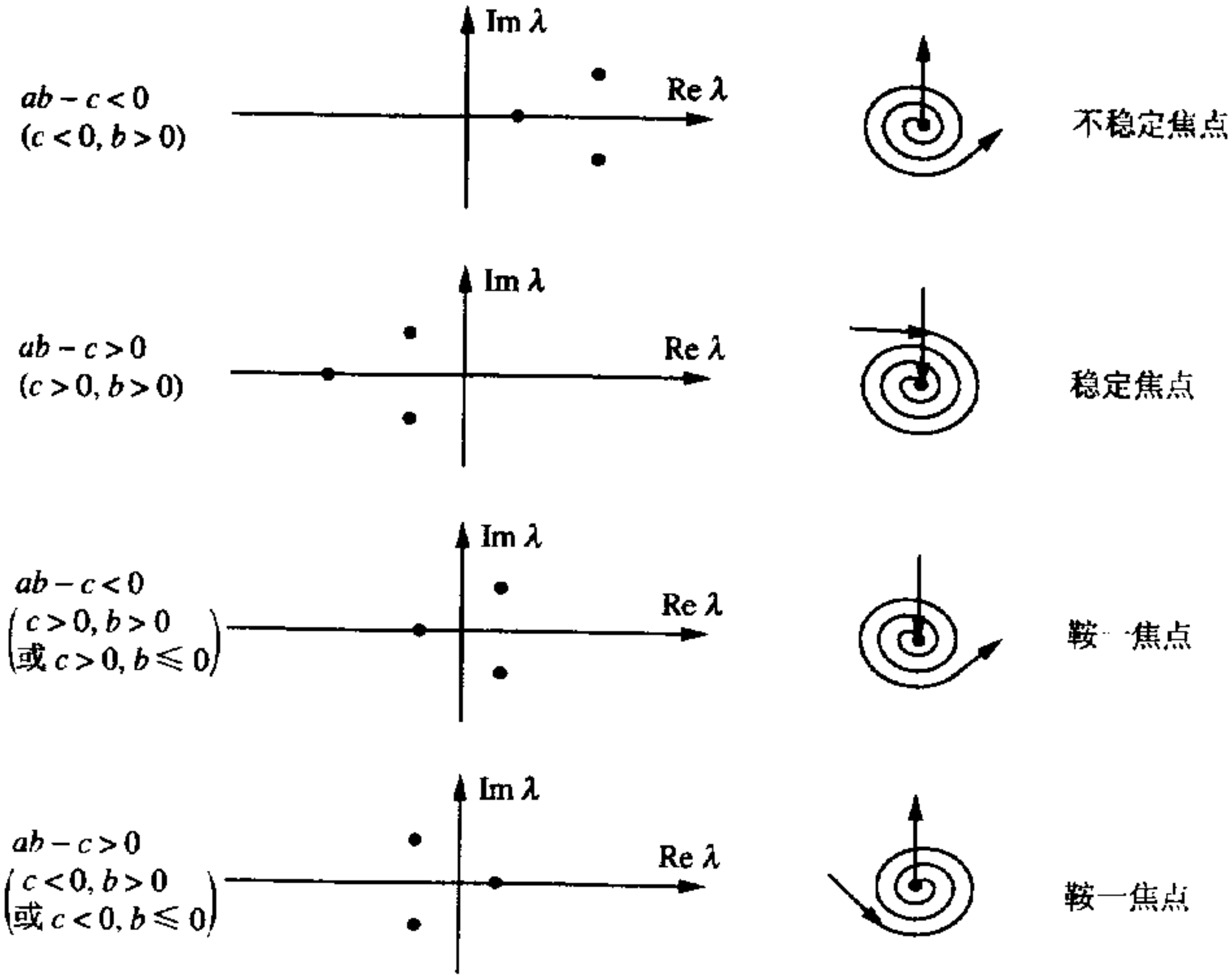
-

$$\Delta = -a^2b^2 + 4b^3 + 4a^3c - 18abc + 27c^2 < 0$$

-


Im λ
Re λ
ab − c < 0
(c < 0, b > 0)
不稳定结点
ab − c > 0
(c > 0, b > 0)
稳定结点
ab − c < 0
(c > 0, b > 0
或 c > 0, b ≤ 0)
鞍点
ab − c > 0
(c < 0, b > 0
或 c < 0, b ≤ 0)
鞍点

- $\Delta > 0$



ab − c < 0
(c < 0, b > 0)
Im λ
Re λ
不稳定焦点
ab − c > 0
(c > 0, b > 0)
Im λ
Re λ
稳定焦点
ab − c < 0
(c > 0, b > 0
或 c > 0, b ≤ 0)
Im λ
Re λ
鞍一焦点
ab − c > 0
(c < 0, b > 0
或 c < 0, b ≤ 0)
Im λ
Re λ
鞍一焦点

•

•

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -az - by - cx \end{cases}$$

• ***$div\ v=-a=\lambda_1+\lambda_2+\lambda_3$*** ***$a=0$***

$\Delta=4b^3+27c^2>0$ -

• -

( )

•

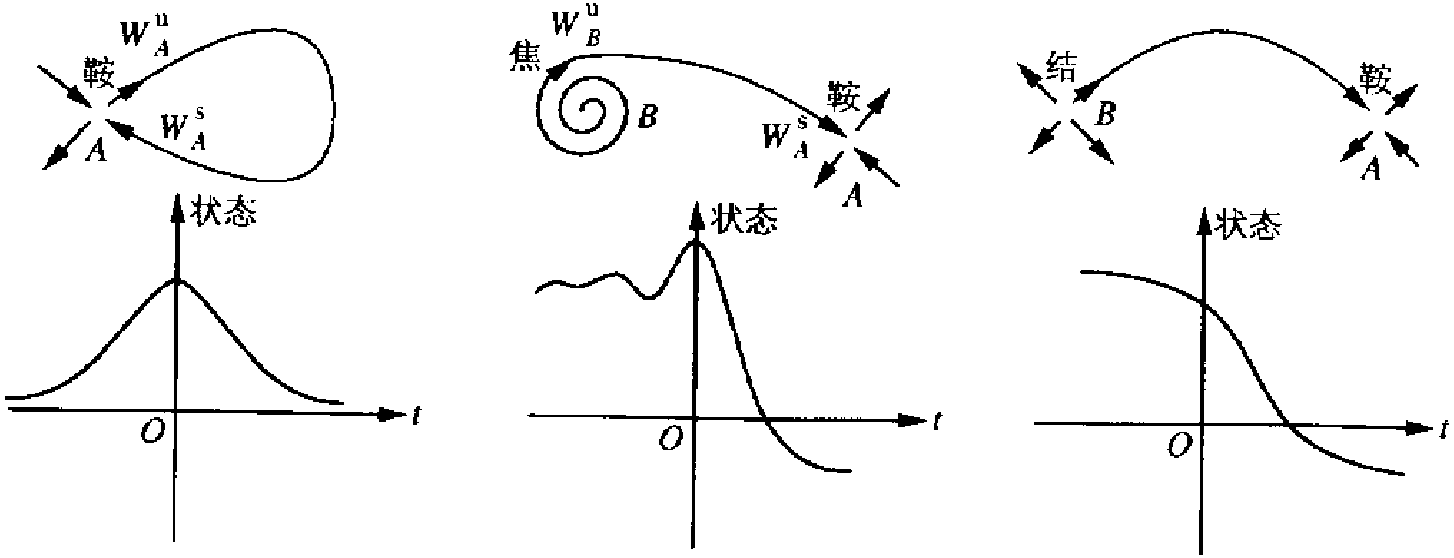
•

(pattern) /

• (homoclinic orbit) $t \to \pm\infty$

• $t \to +\infty$ α $t \to -\infty$ ω

• $t \to +\infty$ $t \to -\infty$ (heteroclinic)

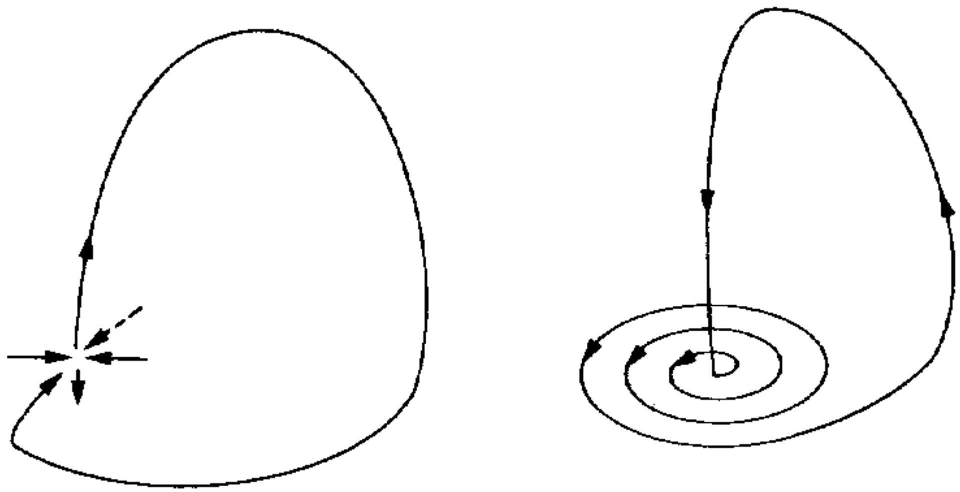


(a) 鞍点同宿轨道及对应的孤立波

(b) 鞍—焦异宿轨道及对应的冲击波

(c) 鞍—结异宿轨道及对应的振荡冲击波

/



(a) 鞍点同宿轨道

(b) 鞍—焦同宿轨道

-
-
- Lorentz
- *Pr* *Ra* *b* Prantdl Rayleigh

$$\begin{cases} \frac{dx}{d\tau} = -Pr\,x + Pr\,y \\ \frac{dy}{d\tau} = Rax + -y - xz \\ \frac{dz}{d\tau} = -bz + xy \end{cases}$$

- (*x*, *y*, *z*)

- $$div\vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -Pr - 1 - b < 0$$
- ***-Prx, -y, -bz*** ***Rax*** **(** **)**

$$\ddot{x} = -Pr\,\dot{x} + Pr\,Rax \Rightarrow \ddot{x} + Pr\,\dot{x} - Pr\,Rax = 0$$

- ***xy*** ***xz*** **Lorentz**

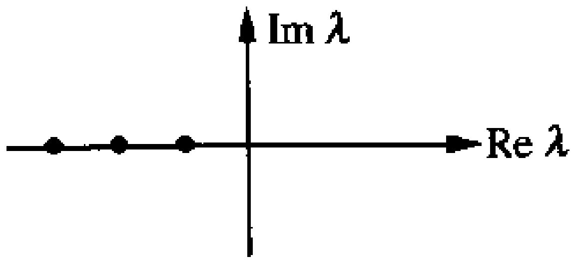
Ra

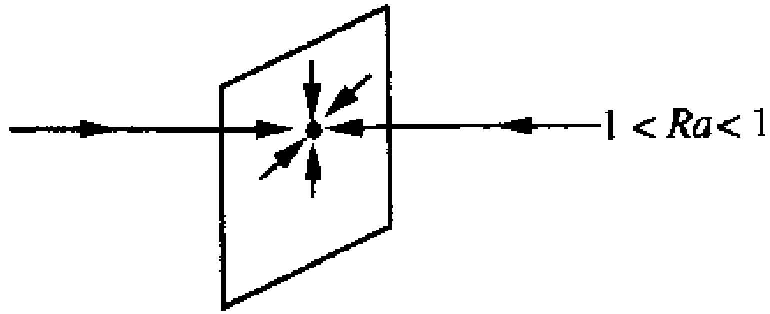
-

$$O: \ (x, y, z) = (0, 0, 0)$$

$$C_{1,2}: \ (x, y, z) = (\pm\sqrt{b(Ra-1)}, \ \pm\sqrt{b(Ra-1)}, \ Ra-1)$$

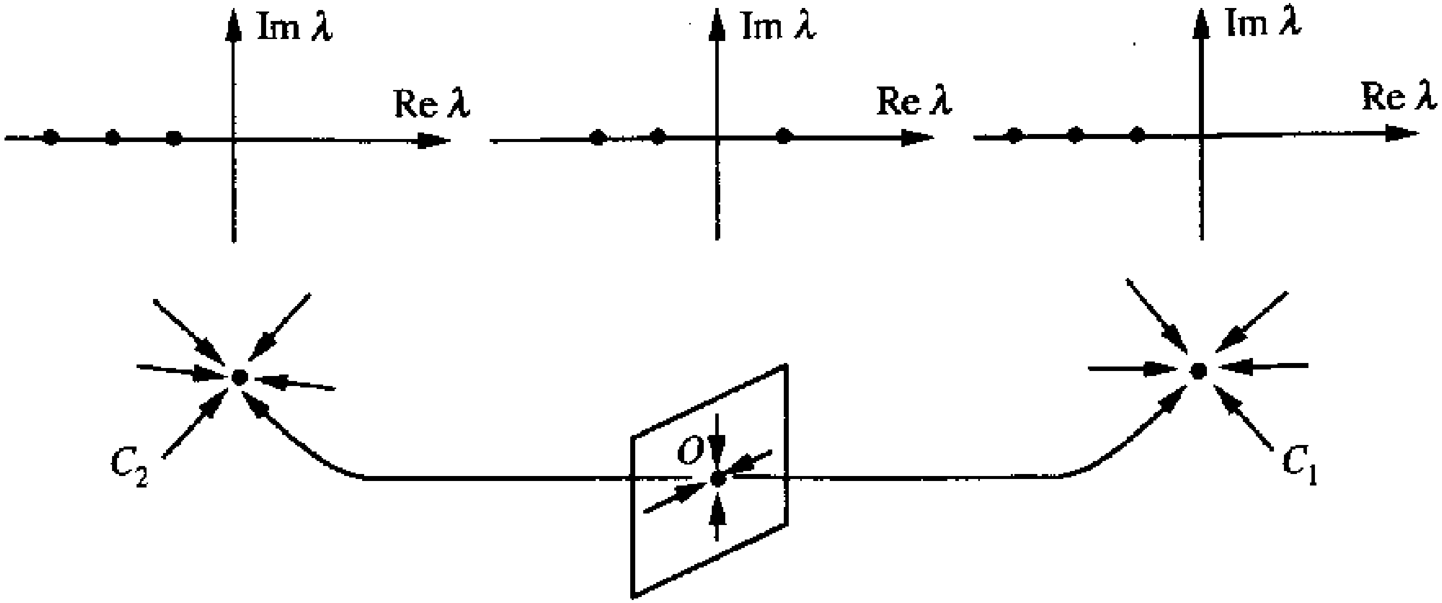
• b=8/3 Pr=10 0≤Ra<1 O $C_{1,2}$





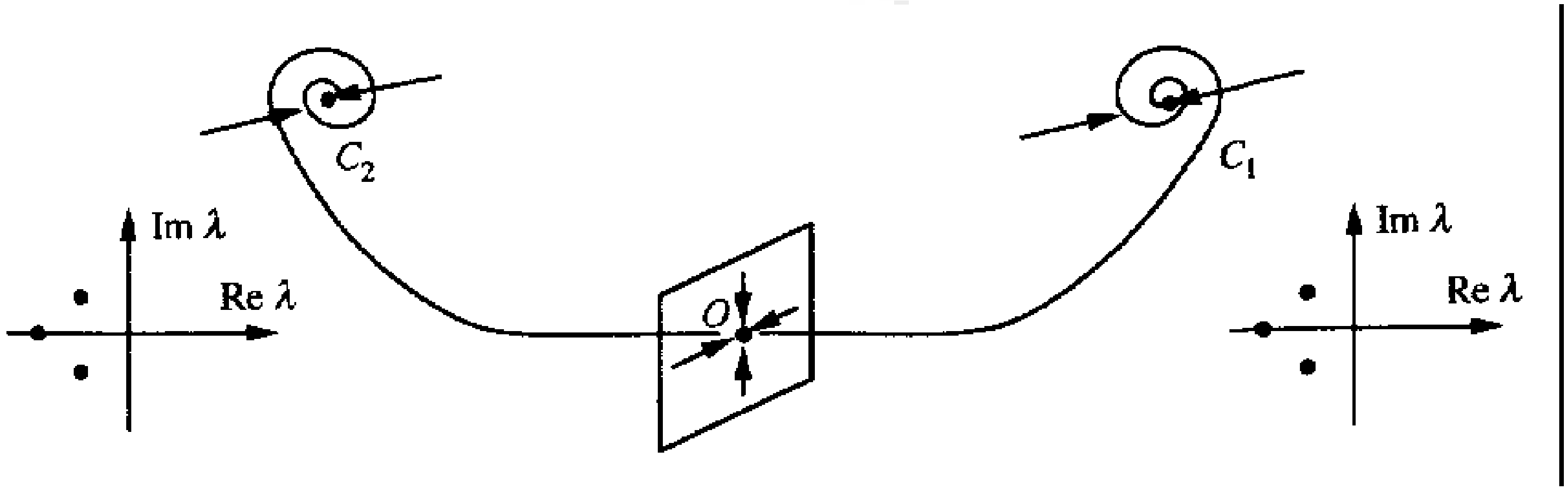
- $1<Ra<1.346$ O $C_{1,2}$ O C_1

  O C_2 -

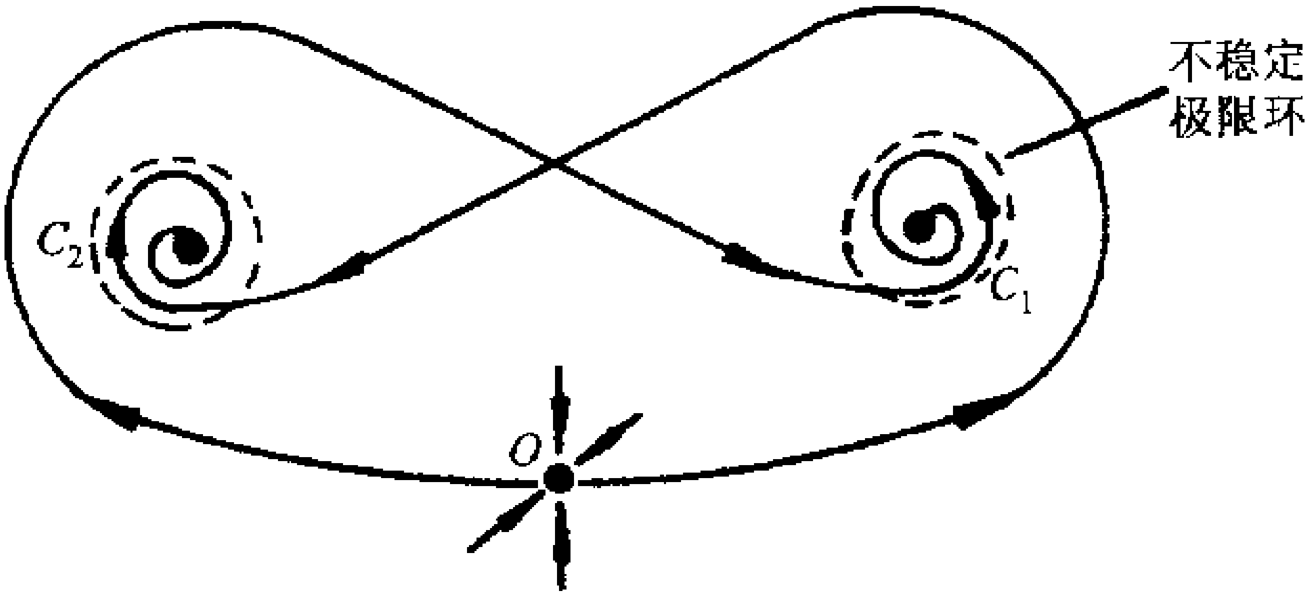


- 1.346<*Ra*<13.926 $C_{1,2}$ O C_1 O C_2

  \- C_1 C_2



- **13.926<*Ra*<24.74 $C_{1,2}$ O C_1 O C_2**



- *Ra*>24.74 $C_{1,2}$ - O C_1

  O C_2