Hysteresis loop area of the Ising model

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The hysteresis of the Ising model in a spatially homogeneous ac field is studied using both mean-field calculations and two-dimensional Monte Carlo simulations. The frequency dispersion and the temperature dependence of the hysteresis loop area are studied in relation to the dynamic symmetry loss. The dynamic mechanisms may be different when the hysteresis loops are symmetric or asymmetric, and they can lead to a double-peak frequency dispersion and qualitatively different temperature dependence.

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When a cooperative many-body system, such as a magnet, is placed in an oscillating external perturbation (such as a magnetic field), it may also show oscillating dynamic response. This response usually lags in time, creating a hysteresis loop with a nonzero area. This phenomenon exists widely in, e.g., magnetic systems and ferroelectric systems,[1] and has been arousing great interest for its important technical application and intriguing physics.[2–4]

The recent theoretical[5–13] and experimental[14] studies on the hysteresis (also, see Ref. 4 and references therein) focus on two topics: the dynamic symmetry breaking and the area of the hysteresis loop. The first phenomenon is due to the competing time scales in such nonequilibrium systems:[4] The hysteresis loop loses its symmetry when the time period of the oscillating external perturbation becomes much smaller than the typical relaxation time of the system. On the other hand, the interesting variance of the hysteresis loop area with such parameters as temperature and oscillation frequency can also be attributed to the time scale competition. For example, in the frequency dispersion of the loop area of the Ising model, the frequency $\omega_0/2\pi$ giving the maximal area corresponds roughly to the point where a resonance occurs. When an Ising system is placed in an ac field, the dynamics may consist of domain nucleation and/or domain growth.[3] The nucleation rate of new domains can be predicted by a characteristic time τ_n, and the domain growth rate is also linked with a characteristic time τ_g. The resonance occurs when the time period of the external perturbation is comparable to either one of these time scales or a combination of them. As is shown below, the details of this dynamic time scale competition necessarily rely on the dynamic phase of the system.

In the present work, we hope to help clarify the relationship of the two above-mentioned topics in the framework of the Ising model, with mean-field (MF) calculations and two-dimensional Monte Carlo (MC) simulations. (The frequency range that receives the most attention here is within the discussion of the previous works using the same methods.[4]) When the loops are symmetric, the system dynamics is controlled by a domain nucleation-and-growth mechanism. When the loops are asymmetric (especially when the magnetization is well above or below zero), throughout the system evolution we can observe most spins being in the same direction. The dynamics of the remaining spins in the opposite direction may be described mainly by the domain nucleation mechanism. In the following, we shall see that the variance of the loop area with frequency, field amplitude, and temperature strongly depends on the dynamic mechanism, which is determined by the loop symmetry.

The model. Before the results are presented, we first describe the model. (1) The evolution of the magnetization M in the mean-field Ising model is determined by the following equation:[4]

$$\frac{dM}{dt} = -M + \tanh\left(\frac{M + H(t)}{T}\right). \tag{1}$$

(2) In the MC simulation, the Hamiltonian of the two-dimensional Ising model in a spatially homogeneous field $H(t)$ can be written as

$$\mathcal{H}(\{\sigma_i\};t) = -J\sum_{\langle i,j\rangle}\sigma_i\sigma_j - H(t)\sum_i \sigma_i. \tag{2}$$

The magnetization is obtained as $M=\Sigma_i\langle\sigma_i\rangle$. The MC simulation goes as follows: The coupling constant J and the Boltzmann constant k_B are both taken as 1. On a two-dimensional $N\times N$ (in the present work, 100×100) lattice with periodic boundary condition, at each time step a spin σ_i is randomly chosen and the probability that it is flipped is[15]

$$W(\sigma_i \to \hat{\sigma}_i) = \frac{1}{Q}\exp\left[-\frac{1}{T}\mathcal{H}(\{\hat{\sigma}_i,\sigma_{j\neq i}\};t)\right], \tag{3}$$

where $\hat{\sigma}_i = \pm\sigma_i$ and Q is the normalization factor. Then, in the next time step the field is updated and another spin is picked at random. A MC step consists of $N\times N$ such unit procedures. The unit time is chosen to be one MC step, with the time resolution being $1/N^2$. The initial state is always 80% randomly chosen spins up (MC simulation) or $M=1$ (MF calculations). In both studies, the system evolves into equilibrium after a zero-field relaxation. Then, an ac field, $H(t) = H_0 \sin \omega t$, is applied. The measurement of the hysteresis loop always begins after a number of introductory cycles, and the symmetry of the loop can be characterized by the order parameter $Q=\int_0^{2\pi/\omega}M dt$.

The phase diagram of the Ising model in an oscillating field has been extensively studied.[4,12,16,17] With regards to the area scaling of the hysteresis loops of the Ising model, there are already MC and MF results, and a detailed review can be

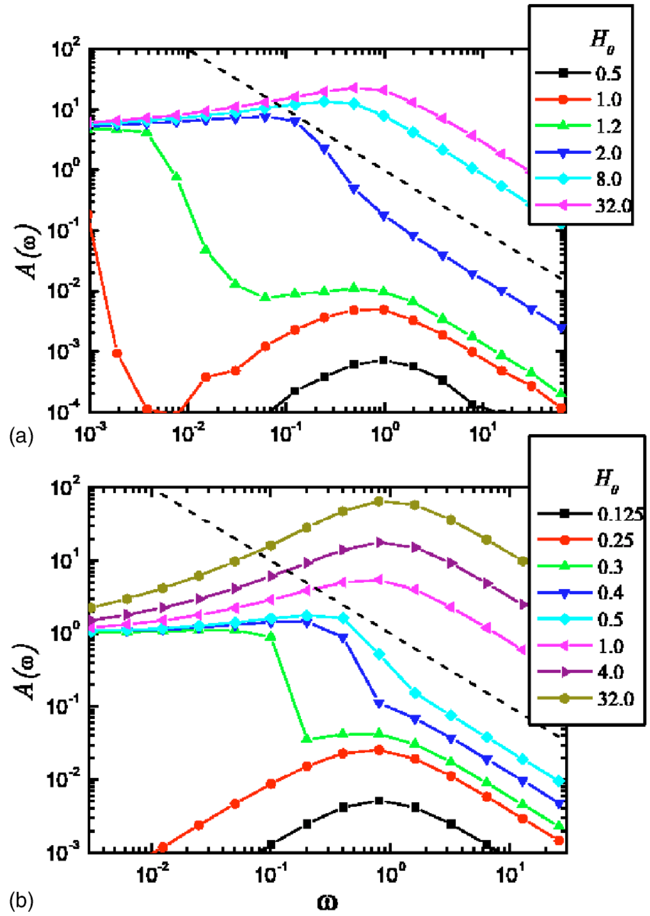

FIG. 1. (Color online) The frequency dispersion of the loop area $A(\omega)$. (a) The MC results with ac field at temperature $T=1.0$; (b) The MF results with ac field at $T=0.5$. The dashed line, corresponding to ω^{-1}, serves as a guide to eye.

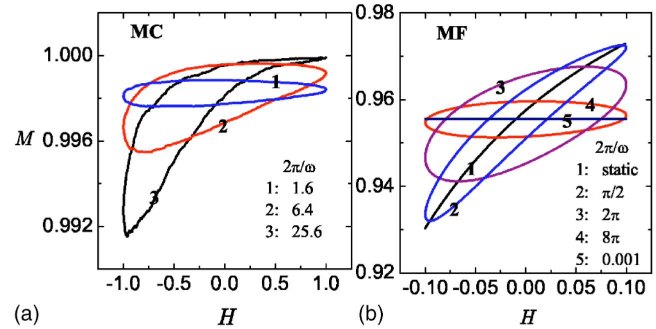

FIG. 2. (Color online) The illustrations of peak II with (a) MC simulations at $T=1.0$ and (b) MF calculations at $T=0.5$.

found in Ref. 4. Generally, the area can be written as $A=A_{\text{sta}}+A_{\text{kin}}$, where A_{sta} is the possible nonzero static contribution existing even in the quasistatic limit, and A_{kin} is the kinetic contribution assuming the form of

$$A_{\text{kin}} \sim H_0^{\alpha} T^{-\beta} g[\tilde{\omega}(\omega, H_0, T)]. \tag{4}$$

Here, $g(\tilde{\omega})$ has been believed to be a single-peak function and $\tilde{\omega}(\omega, H_0, T)$ has been supposed to have the form of $\omega/(H_0^{\gamma}T^{\delta})$.[20] There have been extensive theoretical efforts to determine the exact form of $g(\tilde{\omega})$, and the values and the physical meaning of the exponents. As we shall see below, the dynamic phase transition may lead to a double-peak frequency dispersion and a piecewise analytic function of temperature dependence.

The frequency dispersion of the hysteresis loop area. Figure 1(a) shows a typical result of the two-dimensional MC simulations, at temperature $T=1.0<T_c$. (When $T<T_c$, a dynamic symmetry loss can be observed as H_0 decreases from 4 to 0.[4]) As is clearly shown in the curve with $H_0<2$, two distinct peaks can be observed. We name the left-hand side one as peak I and the right-hand side one as peak II. It is found that peak I occurs in the range where the hysteresis loops are symmetric and peak II is in the range of asymmetric loops. As the frequency is increased from peak I to peak II, a dynamic symmetry loss occurs.

Figure 1(b) shows a typical result of the MF calculations, at temperature $T=0.5<T_c$. Similarly, a double-peak function can be observed. However, there is a very important difference. Given relatively small values of H_0, it is possible that there is only peak II in the MF calculation. This is because the equilibrium magnetization can be obtained by solving the following equation:

$$-M + \tanh\left(\frac{M+H}{T}\right) = 0. \tag{5}$$

With H small enough and $T<T_c=1$, there can be two stable solutions to Eq. (5), corresponding to two values of stable equilibrium magnetization (a positive and negative one). This means that the hysteresis loops can be asymmetric even in the quasistatic limit. By contrast, according to Ref. 4, in MC simulations the hysteresis loop is always symmetric in the quasistatic limit, as a result of fluctuation.

In the above discussions we provide evidence of two peaks I and II, which corresponds to the resonance of symmetric and asymmetric hysteresis loops, respectively. In the following, we give a general explanation of the physics origin of this observation. The existence of two peaks clearly indicates two time scales, corresponding to two different dynamic mechanisms. When the loops are symmetric, both the initial domain nucleation and the late stage domain growth are at work. As suggested by Liu *et al.*,[8,9] the observation of peak I means that a third time scale τ_{I} can be defined as a combination of τ_n and τ_g, and the resonance occurs as $2\pi/\omega_{\text{I}} \sim \tau_{\text{I}}$. Since for the asymmetric loops the magnetization can be always well above or below zero, at any time during the system evolution we can observe most spins having the same direction. The late stage domain growth is relatively inhibited, and thus, the time scale τ_{II} corresponding to peak II shall be mainly determined by τ_n.

Compared with peak II, the resonance at peak I has been relatively well studied in the previous works. Some illustrations of the resonance at peak I can be found in, e.g., Refs. 9 and 11. In the following we focus on peak II, as illustrated in Figs. 2(a) and 2(b). Actually, the existence of peak II can be easily understood in the MF Ising model. As is mentioned above, when H is small enough, there can be two stable solutions of Eq. (5). Thus, with $\omega \to 0$, the hysteresis loop

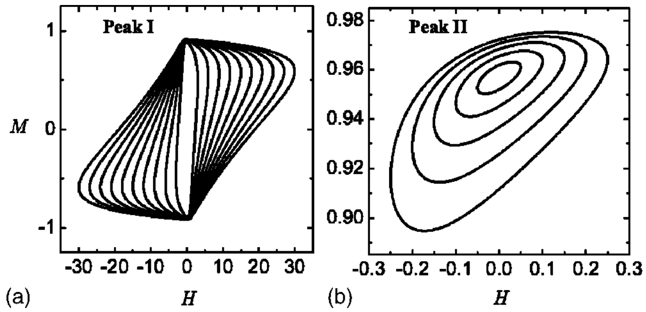

FIG. 3. (a) Some typical hysteresis loops at peak I, with $H_0=3,6,\ldots,30$ from the innermost to the outermost loop. (b) Some typical hysteresis loops at peak II, with $H_0=0.05,0.15,\ldots,0.25$. All the results are obtained from MF calculations at $T=0.5$.

reduces to a curve [the stable solution of Eq. (5), as shown in Fig. 2(b)]. In the other limit, with $\omega\to\infty$, the system cannot respond to the external field and the hysteresis loop becomes a horizontal line. Thus, it is straightforward to predict the existence of a peak in the intermediate region, where the phase lag between the external field and the system response creates a hysteresis loop with a nonzero area. Note that the observation of peak II requires temperatures lower than the static critical point and small enough values of H_0. When $H_0 \ll T \ll 1$, we can give a quantitative description of the system behavior by solving the MF equation (1) analytically. We suppose $1-M\to 0$, and obtain

$$\frac{dM}{dt} = 1 - M - 2\exp\left(-\frac{2}{T}\right)\left(1-\frac{2}{T}H_0\sin(\omega t)\right).$$

This equation can be exactly solved, and the loop area is obtained as

$$A = -\int_0^{2\pi/\omega} M d(H_0 \sin \omega t),$$

$$=4\pi\left[\frac{1}{T}\exp\left(-\frac{2}{T}\right)\right]H_0^2\frac{\omega}{\omega^2+1}. \qquad (6)$$

In the following, we report some important differences and similarities of peak I and II, as summarized from Eq. (6) and the numerical results in Fig. 1. Differences: (1) The temperature dependence of the height of peak II in the MF Ising model is obtained in Eq. (6), and is different from the previous $T^{-1/2}$ prediction of peak I.[17] (2) In the MF Ising model, the maximal area $A_{\max}^{\mathrm{II}}$ at peak II grows with H_0 as H_0^2, while the maximal area $A_{\max}^{\mathrm{I}}$ at peak I has been predicted to grow with H_0 linearly in the previous studies.[17] This difference is explained by observing the variance of the loop shape with H_0, as illustrated in Figs. 3(a) and 3(b): At peak I, only the width of the loop increases with H_0 (that is why $A_{\max}^{\mathrm{I}}$ grows linearly with H_0), while at peak II, the loop is expanding in two directions with $A_{\max}^{\mathrm{II}}$ growing as H_0^2.

Similarities: (1) In both MF calculations and MC simulations, it is found that, as $\omega\to\infty$, the area decays as ω^{-1}. [With respect to peak I, this is in accordance with the previous MF result and the work of Rao *et al.*[18,19] on the $(\Phi^2)^2$ and $(\Phi^2)^3$ model, but not the previous MC result of the Ising model,[4,17]

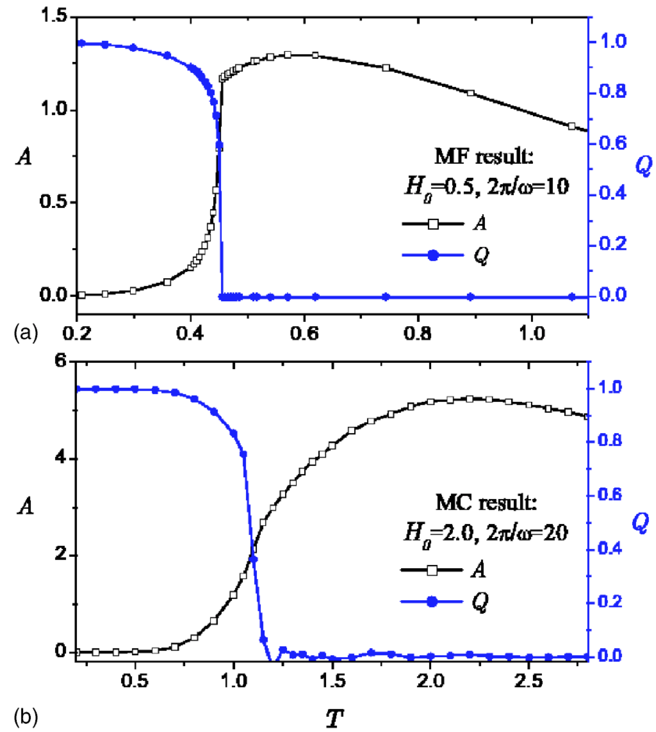

FIG. 4. (Color online) The temperature dependence of the loop area A and the order parameter Q obtained from (a) MF and (b) MC calculations, with the ac field.

which indicates an exponentially decaying function of $g[\tilde{\omega}(\omega,H_0,T)]$ in Eq. (4).] (2) Independent of H_0 and T, peak II is always observed to be at (or very close to) $\omega_{\mathrm{II}}=1$. With regards to peak I, in both MC simulations and MF calculations we find that as $H_0\to\infty$, ω_{I} also approaches 1, which is different from the previous predictions. According to Refs. 4,17 the ω_{I} will also tend to infinity as $H_0\to\infty$. But it is not what we observe in Fig. 1, which shows that, as the field already far exceeds the spin-spin interaction, the time scale of the system is no longer sensitive to the value of H_0.

In the following, we turn to study *the temperature dependence of the loop area* with fixed field amplitude and frequency.[13,17] Here, our motivation is quite similar to that of the above discussions of the frequency dispersion. When $H_0<4$ (MC) or $H_0<1$ (MF), a dynamic symmetry loss can be observed as T decreases (as can be predicted from the phase diagram previously obtained).[4,12,16,17] Thus, a simple scaling function is not likely to exist for the loop area, since there are different dynamic mechanisms of the symmetric and asymmetric loops, and different time scale competition. This is supported by the MF and MC results.

A typical MF result is shown in Fig. 4(a). The order parameter $|Q|>0$ for lower temperature and $|Q|=0$ for higher temperature, and at the dynamic critical point, a dynamic symmetry loss occurs. In Ref. 13, using the same methods as the present work, Acharyya found that the area becomes maximum above the dynamic transition point. Here, it is clear that the temperature dependence of the loop area assumes different functions for the $|Q|>0$ states and $|Q|=0$ states. These different functions are separated by the dynamic critical point and the first-order derivative, $\partial A/\partial T$, is not continuous at the dynamic critical point. Thus, the temperature dependence is a piecewise analytic function. A typical MC result is shown in Fig. 4(b), and it is roughly similar to the MF result. Although we do not observe a notable discontinuity of the first-order derivative, $\partial A/\partial T$, it is very obvious that the second-order derivative, $\partial^2 A/\partial T^2$, changes sign at the dynamic critical point. When $H_0 > 4$ (MC) or $H_0 > 1$ (MF), the field amplitude exceeds the spin-spin interaction and the hysteresis loops are always symmetric.[4,12,16,17] In this case, we observe that the loop area decreases monotonically as the temperature grows.

To summarize, the hysteresis of the Ising model in an ac field, $H(t)=H_0\sin(\omega t)$ is studied using both mean-field calculations and two-dimensional Monte Carlo simulations. The frequency dispersion and the temperature dependence of the loop area are studied in relation to the dynamic symmetry loss. The dynamic mechanisms are different when the hysteresis loops are symmetric or asymmetric. For symmetric loops, the dynamics is a combined domain nucleation-and-growth process. By contrast, for asymmetric loops well above or below the $M=0$ line, the dynamics may be mainly domain nucleation. This framework is part of basic current knowledge of hysteresis phenomena, and the observed frequency and temperature dependence of the loop area is consistent with it. Double peaks can be observed in the frequency dispersion, and the temperature dependence is possibly a piecewise analytic function. Interestingly, the shift of the dynamic mechanism with the symmetry loss is also found in the mean-field calculation, and some striking similarities are observed (for example, the same position of peak II). Although the present work deals with a model spin system, the topics discussed have general meaning. Surely some quantitative details, like the position of the peaks, rely on the model setting, but we believe that the physics of the conclusions is not limited to the specific system studied here, and can be predicted for more general systems.

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